Relativistic Mechanics: Mass-energy Relation

Recap: in pre-relativistic physics the following situation holds:

(1) The space and time coordinates transform according to the "Galilean" prescription

$$x' = x - vt,\ t' = t.$$

As a result, velocities obey the "common-sense" composition law

$$v_{A-C} = v_{A-B} + v_{B-C}$$

(2) The total mass of a closed system is conserved. (This is so "obvious" that it is rarely stated!)

(3) As a result, Newtonian *mechanics* is invariant under Galilean transformations. In particular if the momentum of a closed system is conserved in one inertial frame it is conserved in any: if $P = \sum_i m_i v_i =$ const., then in frame S'

$$P' = \sum_i m_i v_i' = \sum_i m_i(v_i - u) = P - u\sum_i m_i = P - uM$$

and hence P' =const. by (2).

(4) The laws of electromagnetism (optics) are *not* invariant under transformation to a different inertial frame (hence the need to postulate the "ether").

Einstein "corrected" (4) by postulating that a shift from one inertial frame to another is implemented not by a Galilean but by a Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}},\ t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \qquad (1')$$

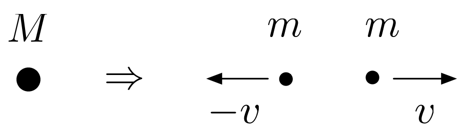
As a result, velocities do *not* satisfy the "common-sense" composition law, but rather (lecture 12):

$$v_{A-C} = \frac{v_{A-B} + v_{B-C}}{1 + v_{A-B}v_{B-C}/c^2}$$

With this transformation law, the laws of electromagnetism (optics) look the same to any inertial observer.

Now, if this is so, it is very much in the spirit of special relativity to say: *All* the laws of physics look the same to any inertial observer (as was true for *mechanics* in pre-relativistic theory). But is this true for mechanics? I.e. is mechanics in fact Lorentz-invariant?

A simple thought-experiment shows that we cannot, in special relativity, simultaneously maintain (a) the conservation of mass, (b) Newton's laws with the standard definition of momentum, and (c) general Lorentz invariance. Imagine a particle of mass M which disintegrates, e.g. by a radioactive process, into 2 particles of mass m By (a),



we must evidently have $M = 2m$. Consider now the conservation of momentum as seen by (a) an observer O at rest with respect to M, (b) an observer O' moving relative to M with *velocity* $+u$.

According to O, the original momentum is zero and hence, by Newton's second and third laws, so must be the final momentum: since $P = m_1 u_1 + m_2 v_2$ and $m_1 = m_2$, this implies $v_1 = -v_2 = v$ as in the diagram. According to O', the original momentum is $-Mu = -2mu$. What is the final momentum? According to the relativistic law of composition of velocities, he reckons

$$v_1' = \frac{-u + v_1}{1 - uv_1/c^2} = \frac{-u + v}{1 - uv/c^2} \text{ and } v_2' = \frac{-u + v_2}{1 - uv_2/c^2} = \frac{-u - v}{1 + uv/c^2}$$

Thus, the total final momentum is

$$P'_{(f)} = m[\frac{-u + v}{1 - uv/c^2} + \frac{-u - v}{1 + uv/c^2}] = -2mu[\frac{1 - v^2/c^2}{1 - (uv)^2/c^4}]$$

This is *different* from the original momentum $P'_{(i)} = -2mu$ seen by O' (note that u can never be c in special relativity!). Thus, if we stick to the Newtonian definition of momentum, and assume conservation of mass, mechanics *cannot* be Lorentz-invariant. We are then faced with the choice

(a) live with this situation-but then the whole point of special relativity goes out the window.

(b) modify one or more principles of Newtonian mechanics.

One might perhaps think that the way to fix up the problem would be to relax the assumption $M = 2m$. But it is clear that this *alone* cannot work, since we would have to postulate $M/2m = (1 - v^2/c^2)/(1 - (uv)^2/c^4)$,which depends on the velocity of O', and hence would violate Lorentz invariance. Another way of saying this is to note that we can certainly do collision experiments in which the outgoing particles are *the same* as the ingoing ones, so that there can be no question of mass change: generalizing the above argument to this case, then two observers O and O', moving with a velocity u with respect to one another, who each apply Newton's second and third laws, will reckon as follows:

$$O \text{ reckons:} \quad P_{in} = \sum_i m_i v_{in}^{(i)} = \sum_i m_i v_{out}^{(i)} \equiv P_{out}$$

On the other hand

$$O' \text{ reckons:} \quad P'_{in} = \sum_i m_i v_{in}'^{(i)} = \sum_i m_i \frac{(v_{in}^{(i)} - u)}{1 + uv^i_{(in)}/c^2}$$

$$P'_{out} = \sum_i m_i v'^{(i)}_{out} = \sum_i m_i \frac{(v^{(i)}_{out} - u)}{1 + uv^i_{(out)}/c^2}$$

and, in general, if $P_{in} = P_{out}$ then we cannot simultaneously have $P'_{in} = P'_{out}$, so that it is impossible for both O and O' to be right.

Let's try to sum up the situation. In Newtonian physics, the momentum of a system is defined as its mass times its velocity, and the combination of N2 and N3 implies that the total momentum of a closed system on which no external forces act is conserved. This statement is invariant under Galilean transformation. However, if in accordance with the precepts of special relativity we replace the Galilean transformation by a Lorentz one, then the statement is no longer invariant, i.e. if one inertial observer sees momentum to be conserved then in general others will not. At this stage we have two obvious alternative strategies:

(a) keep the Newtonian definition of momentum, but accept that N2 and N3 (and hence the law of conservation of total momentum) are valid only in a particular frame of reference, or

(b) modify the definition of momentum so that N2 and N3, and hence the conservation of total momentum, is valid for all inertial observers.

Faced with this dilemma, why do most physicists (at least nowadays) unhesitatingly plump for (b)? Probably for much the same kind of reasons as special relativity is almost universally preferred to the "contraction" theories of Lorentz and Fitzgerald: the ultimate outcome in terms of experimental predictions is exactly the same, but the theory which results from a full-blooded acceptance of the *complete* equivalence of all inertial frames, that is, equivalence for the purposes of mechanics as well as optics, is so much simpler and (in a sense which is easy to recognize but difficult to define) more "elegant" than the alternative that there is, in most people's minds, really no contest.

So it is apparently necessary to change the definition of momentum, if Newton's second and third laws are to continue to hold.

Some clues:

(1) The definition should reduce to mv for $v \to 0\,(v \ll c)$.

(2) Under the action of a constant force (if Newton's second law is to continue to hold) p increases indefinitely, but we want u never to reach c. Thus the relation between p and u should be such that $p = \infty$ for $v = c$.

A possible conjecture is:

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

Does this enable us to satisfy simultaneously Newton's second and third laws, and Lorentz invariance?

Suppose we *assume* that the total momentum, defined in this way, is conserved for inertial observer O, i.e.

$$P = \sum_i \frac{m_i v_i}{\sqrt{1 - v_i^2/c^2}} = \text{const.}$$

(i.e. unchanged when the final values of v_i, are substituted for the initial ones)

What then is the conclusion of the inertial observer O' who moves with the velocity (say) u relative to O? He will see a total momentum given by

$$P' = \sum_i \frac{m_i v_i'}{\sqrt{1 - v_i^2/c^2}}$$

But according to the relativistic law of addition of velocities,

$$v_i' = \frac{v_i - u}{(1 - uv/c^2)}$$

and therefore

$$P' = \sum_i m_i \left(\frac{(v_i - u)/(1 - uv_i/c^2)}{\sqrt{1 - \frac{(v_i - u)^2/c^2)}{(1 - uv_i/c^2)^2}}} \right)$$

and after a bit of algebra (see appendix) this becomes:

$$P' = \sum_i \frac{m_i(v_i - u)}{\sqrt{1 - v^2/c^2}} \times \frac{1}{\sqrt{1 - u^2/c^2}}$$

$$\equiv \frac{1}{\sqrt{1 - u^2/c^2}} \sum_i \frac{m_i v_i}{\sqrt{1 - v_i^2/c^2}} - \frac{u/c^2}{\sqrt{1 - u^2/c^2}} \sum_i \frac{m_i c^2}{\sqrt{1 - v_i^2/c^2}} \leftarrow (c^2 \text{ added for convencience!})$$

Now, the first term is simply $(1 - u^2/c^2)^{-1/2} \times P$, so if P is conserved so is this term. So we see that if the conservation of P is to imply that of P' we must have

$$\sum_i \frac{m_i c^2}{\sqrt{1 - v_i^2/c^2}} = \text{const.} \qquad (*)$$

in any collision process in a closed system of particles.

What is the significance of this statement? Let's assume all velocities are $\ll c$, and expand:

$$\sum_i m_i c^2 (1 + \frac{1}{2} v_i^2/c^2 + \ldots o(v^4)) = \text{const.}$$

Now the first term is just $c^2 \times \sum_i m_i$, which we know is unchanged in the collision since the same elementary particles enter and leave (in the case considered). The next term, $\frac{1}{2} \sum_i m_i v_i^2$, is the nonrelativistic kinetic energy, which in nonrelativistic mechanics we know to be conserved for a closed system. Thus, it seems reasonable to regard the expression

$$\frac{mc^2}{\sqrt{1-v^2/c^2}}$$

as the appropriate generalization of the expression for the (non-potential) energy in relativistic mechanics. However, it is unclear so far whether or not the zerothorder term, mc^2, is to be taken seriously. One advantage of doing so is that we can then write the expression for P' in the form

$$P' = \frac{P - uE/c^2}{\sqrt{1-u^2/c^2}}$$

and it also turns out (not proved!) that we also have

$$E' = \frac{E - uP}{\sqrt{1-u^2/c^2}}$$

so that under a Lorentz transformation P and E transform (rather) like x and t. Thus (part of) what is seen as the effects of conservation of momentum in one frame can look like that of conservation of energy in another and vice versa.

A prima facie difficulty: what happens in a situation where, because of some external agency, E is conserved but P is not? (e.g. a ball bouncing against a "fixed" wall, cf. lecture 8). Answer: In general, in a different Lorentz frame *neither* E nor P will be conserved! (The moving wall can transfer energy as well as momentum.) (Note: This consideration is common to relativistic and Newtonian mechanics, cf. Lecture 7.)

However, this "neat" result does not establish that we should really count the mc^2 as part of the total energy. To determine whether we should or not, we need to go back to a case where total mass is not obviously conserved, e.g. the disintegrating particle discussed earlier. We note that the argument that Lorentz invariance requires the conservation of the expression marked $(*)$ goes through unchanged for this case, and thus we have for *any* inertial observer the statement

$$\sum_i \frac{m_i c^2}{\sqrt{1-v_i^2/c^2}} = \text{const.}$$

in any disintegration/collision process. Let's then apply this result in the frame of an observer at rest with respect to the original (about-to-disintegrate) particle of mass M. Assuming that the two decay products have each mass m, and that their velocities are equal and opposite (conservation of momentum!) we then have

$$Mc^2 = \frac{2mc^2}{\sqrt{1-v^2/c^2}}$$

or:

$$M = \frac{2m}{\sqrt{1-v^2/c^2}} \; (\neq 2m!)$$

Thus, the mass of the original (unstable) particle is *greater* than that of the decay products: part of the original mass is turned into kinetic energy of the decay products (which will, of course, be lost when e.g. the latter are slowed to rest in a medium). *Mass* as such is not conserved: total *energy*, including the rest energy mc^2, is! The crucial point is that this is an *experimentally measurable effect*! The "mass" is simply the inertial mass, and this is measured by resistance to acceleration. This effect is very small (typically $\sim 10^{-9}$) for a typical chemical reaction, but may be a good deal more substantial for e.g. the fission of uranium (v/c here $\sim 1/30$, so the mass changes by a factor $\sim 10^{-3}$ of 200 GeV $\sim$ 200 MeV/disintegration).

To summarize the conclusions reached in this lecture: Once we have decided that the proper way to relate the space and time coordinates measured by different inertial observers is through the Lorentz rather than the Galilean transformation, then there is no way of maintaining the invariance of the law of conservation of total momentum, so long as momentum is simply defined as *mass* $\times$ *velocity*(mv). We can maintain the invariance by redefining

$$P = mv/\sqrt{1 - v^2/c^2}$$

but *only* if we assume that the sum of the quantities $mc^2/\sqrt{1 - v^2/c^2}$ is conserved. In particular, for a single particle decaying into two this implies that the sum of the masses of the decay products is in general less than that of the original particle, i.e. the total mass of the system (as conventionally defined) is not conserved. Rather, it is the total *energy*, defined as the sum of the quantities $mc^2/\sqrt{1 - v^2/c^2}$ which is conserved. In the limit $v \to 0$ this is expressed, of course, in the famous relation $E = mc^2$.

It is somewhat ironical that the special theory of relativity is so often regarded as the beginning of "modern physics"; as remarked by Hesse, in many ways it is more natural to regard it as the culmination of classical physics, that is as the final unification of the mechanics of Newton and the electromagnetic, theory of Maxwell. One special relativity is in place, classical physics stands as a self-consistent whole, which "could have" been the Final Theory of the universe. As we shall see, it was not to be ...

Appendix: Algebra for the transformation of momemtum

The total momentum P' as seen by inertial observer O' is given by the expansion

$$P' = \sum_i \frac{m_i(v_i - u)/(1 - uv_i/c^2)}{\sqrt{1 - \frac{(v_i - u)^2/c^2}{(1 - uv_i/c^2)^2}}}$$

Consider the expansion

$$J_i \equiv \frac{(v_i - u)/(1 - uv_i/c^2)}{\sqrt{1 - \frac{(v_i - u)^2/c^2}{(1 - uv_i/c^2)^2}}}$$

Multiply the numerator and denominator by $1 - uv_i/c^2$:

$$J_i = \frac{v_i - u}{\sqrt{1 - uv_i/c^2)^2 - (v_i - u)^2/c^2}}$$

$$= \frac{v_i - u}{\sqrt{1 - 2uv_i/c^2 + u^2v_i^2/c^4 - v_i^2/c^2 + 2uv_i/c^2 - u^2/c^2}}$$

$$= \frac{v_i - u}{\sqrt{1 + u^2v_i^2/c^4 - v_i^2/c^2 - u^2/c^2}}$$

$$= \frac{v_i - u}{\sqrt{(1 - u^2/c^2)(1 - v_i^2/c^2)}} \equiv \frac{v_i - u}{\sqrt{1 - v_i^2/c^2}} \times \frac{1}{\sqrt{1 - u^2/c^2}}$$

Thus,

$$P' \equiv \sum_i m_i J_i = \frac{1}{\sqrt{1 - u^2/c^2}} \cdot \sum_i \frac{m_i(v_i - u)}{\sqrt{1 - v_i^2/c^2}}$$

as stated in the text.